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# THE STRENGTH-DIFFERENTIAL EFFECT IN PLASTICITY

J. CASEY and H. JAHEDMOTLAGH
Department of Mechanical Engineering, University of Houston, 4800 Calhoun, Houston, TX 77004, U.S.A.

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**Abstract**—In the context of a purely mechanical rate-type theory of elastic-plastic materials, a special set of constitutive equations is discussed. In particular, a yield function is chosen which includes a dependence on mean normal stress. The constitutive equations are capable of describing the strength-differential effect and also predict a plastic volume change. The values of the material coefficients in the constitutive equations are determined from published experimental data for AISI 4330 steel. The predicted results are in good agreement with experiments, except for the magnitude of the predicted plastic volume change, which is too large.

## 1. INTRODUCTION

In recent years it has been established by experiment that the yield stress of high-strength steels is appreciably greater in uniaxial compression than in tension.† This variation in yield stress is referred to as the strength-differential or S-D effect. Its main features are:

(i) it persists throughout the entire plastic strain range; and

(ii) it is accompanied by plastic volume expansion in both tension and compression tests.

Due to the first feature, the S-D effect stands in contrast to Bauschinger and allied effects, which become unimportant at large plastic strains. In order to establish the presence of a true S-D effect as distinct from a variation in yield strength that is peculiar to the region of initial yield, it is clearly necessary to continue experiments well into the work-hardening range. Also, in connection with the identification of initial yield, it is important to choose an appropriately large plastic strain offset.‡

Although in the present paper we are concerned only with the S-D effect in high-strength steels, it is perhaps worth mentioning that other materials, including polymers, soils and rocks also exhibit an S-D effect[1]. Of course, for pressures of the order of the yield strength, the S-D effect is negligible for low-strength ductile metals[1].

An S-D effect could conceivably arise from the preferential opening of microcracks in tension tests. However, as noted by Spitzig *et al.* [2], the similarity in the magnitude of the volume expansion under tensile and compressive loading indicates that this mechanism is in fact not the cause of the S-D effect. Rather, the S-D effect is ascribed to the inhibiting influence of pressure on shear-activated slip within grains.§ Mathematically, this is reflected in the dependency of the yield function on mean normal stress. That this dependency is significant for high-strength steels has been established by the experiments of Spitzig *et al.* [2]. If a normality condition for plastic strain rate is satisfied, it then follows that plastic volume change occurs, in qualitative agreement with feature (ii) above.

Spitzig *et al.* [2] concluded that the yield function also depends slightly on the third invariant of the deviatoric stress tensor. However, a re-examination of their data by Gupta[3] has shown that such a conclusion is unwarranted.

Based on the above considerations, in the context of a general theory of plasticity due to Green and Naghdi[4, 5], we utilize in the present paper a yield function that includes a dependence on mean normal stress $\bar{s}$. This dependency is accounted for by means of two

†For an extensive list of references, see the papers by Drucker[1] and Spitzig *et al.* [2].

‡See Table 1 of [1] for recommended values.

§In this regard also the S-D effect is distinct from the Bauschinger effect, which is attributable to residual stresses due to the grain boundaries.

material coefficients, $\psi$ and $\eta$.† Unless $\psi$ is nonzero, there is no S-D effect and no plastic volume change. In order for an S-D effect to be present at zero pressure, it is necessary that $\eta$ be nonzero also.

We now outline the contents of the paper. Section 2 contains a summary of the general theory of plasticity originally proposed by Green and Naghdi[4, 5], and subsequently developed by Naghdi and Trapp[6, 7] and Casey and Naghdi[8, 9]. In particular, a strain space formulation of the theory is adopted, the advantages of which are discussed in [6]. Also, the method of characterizing strain hardening behavior which was proposed in [8, 9] is discussed.

In Section 3 we present a special set of constitutive equations which are capable of describing the S-D effect and also predict a plastic volume change. These equations satisfy a normality condition for plastic strain rate. In Section 4 the experimental data of Spitzig *et al.* [2] is utilized to obtain values for material coefficients that appear in the constitutive equations of Section 3. The S-D effect, plastic volume change and the strain-hardening characteristics of AISI 4330 steel are analyzed. A discussion of these results is contained in Section 5.

## 2. BACKGROUND INFORMATION

Let the motion of a body be referred to a fixed system of rectangular Cartesian coordinate axes and let a typical particle of the body occupy the position $X_A$ in some fixed reference configuration. Further, let $x_i$ designate the position of the particle in the present, or deformed, configuration at time $t$. Then the motion of the body is defined by

$$x_i = \chi_i(X_A, t). \tag{2.1}$$

The deformation gradient relative to the reference position and its determinant are

$$F_{iA} = \frac{\partial \chi_i}{\partial X_A},\ J = \det(F_{iA}) > 0. \tag{2.2}$$

Throughout this paper, indices take the values 1, 2, 3, and the usual convention of summation over repeated indices is employed. Lower case indices are associated with the spatial coordinates $x_i$ and upper case indices refer to material coordinates $X_A$.

We define a symmetric Lagrangian strain tensor $e_{KL}$ by

$$e_{KL} = \frac{1}{2}(F_{iK}F_{iL} - \delta_{KL}), \tag{2.3}$$

where $\delta_{KL}$ is the Kronecker delta. The six-dimensional Euclidean vector space formed from the components of $e_{KL}$ is called *strain space*.

Recall the relationship between the symmetric Piola–Kirchhoff stress tensor $s_{KL}$ and the symmetric Cauchy stress tensor $t_{ij}$, namely

$$Jt_{ij} = F_{iK}F_{jL}s_{KL}. \tag{2.4}$$

The six-dimensional Euclidean space formed from the components of $s_{KL}$ is called *stress space*.

In addition to the strain tensor $e_{KL}$, at each point of the continuum we admit the existence of a plastic strain, specified by a symmetric second order tensor $e^p_{KL}$, and a measure of work-hardening, specified by a scalar-valued function $\kappa$. We also assume the existence of a scalar-valued yield (or loading) function $g$ in strain space such that for fixed values of $e^p_{KL}$ and $\kappa$, the equation

$$g(e_{KL}, e^p_{KL}, \kappa) = 0 \tag{2.5}$$

†See (3.4)$_1$ and also (3.18).

represents a closed orientable hypersurface of dimension five enclosing an elastic region in strain space. The function $g$ is chosen so that $g < 0$ for all points in the elastic region. The hypersurface is called the yield (or loading) surface in strain space.

For the stress response we adopt the special form that was discussed in ([8], eqns 34–42, and Case (b) on p. 291), so that

$$s_{KL} = \bar{s}_{KL}(e_{MN} - e^p_{MN}). \tag{2.6}$$

We also assume the following work postulate due to Naghdi and Trapp[7]: the external work done on an elastic–plastic body in any smooth homogeneous cycle of deformation is nonnegative.

The yield function in stress space is denoted by $f$. For fixed values of $e^p_{KL}$ and $\kappa$, the equation

$$f(s_{KL}, e^p_{KL}, \kappa) = 0 \tag{2.7}$$

represents the five-dimensional yield surface in stress space. The elastic region in stress space corresponds to $f < 0$. The functions $\bar{s}_{KL}$, $g$ and $f$ are taken to be smooth. We recall the notations

$$\hat{g} = \frac{\partial g}{\partial e_{KL}} \dot{e}_{KL}, \ \hat{f} = \frac{\partial f}{\partial s_{KL}} \dot{s}_{KL}, \ \Lambda = \frac{\partial g}{\partial e_{KL}} \frac{\partial f}{\partial s_{KL}}, \tag{2.8}$$

where a superposed dot signifies material time differentiation.

Using the work assumption of Naghdi and Trapp[7], the constitutive equations for the rate of plastic strain and the rate of work-hardening can be expressed in the form[8]:

$$\dot{e}^p_{KL} = \begin{cases} 0 & \text{if } g < 0 & \text{(a)} \\ 0 & \text{if } g = 0 \text{ and } \hat{g} < 0 & \text{(b)} \\ 0 & \text{if } g = 0 \text{ and } \hat{g} = 0 & \text{(c)} \\ \lambda \gamma^* \hat{g} \dfrac{\partial f}{\partial s_{KL}} & \text{if } g = 0 \text{ and } \hat{g} > 0 & \text{(d)} \end{cases} \tag{2.9}$$

and

$$\dot{\kappa} = \xi_{KL} \dot{e}^p_{KL}, \tag{2.10}$$

where $\xi_{KL}$ is a symmetric tensor-valued function and $\lambda$ and $\gamma^*$ are positive scalar-valued functions of the variables $s_{MN}$, $e^p_{MN}$, $\kappa$. The conditions involving $g$ and $\hat{g}$ in (2.9) are the loading criteria of the strain space formulation. These conditions correspond respectively to an elastic state (or point in strain space); unloading from an elastic–plastic state; neutral loading from an elastic–plastic state; and loading from an elastic–plastic state. It is stipulated that $\dot{g} = 0$ during loading. It then follows with the help of (2.9) and (2.10) that

$$1 + \lambda \gamma^* \frac{\partial f}{\partial s_{KL}} \left( \frac{\partial g}{\partial e^p_{KL}} + \frac{\partial g}{\partial \kappa} \xi_{KL} \right) = 0. \tag{2.11}$$

As shown in [8, 9], the strain-hardening behavior of an elastic–plastic material may be characterized by a rate-independent, dimensionless function $\Phi$ which for the constitutive equations of the present paper is given by

$$\Phi = \lambda \gamma^* \Gamma = \frac{\Gamma}{\Gamma + \Lambda}, \tag{2.12}$$

where

$$\Gamma = - \frac{\partial f}{\partial s_{KL}} \left\{ \frac{\partial f}{\partial e^p_{KL}} + \frac{\partial f}{\partial \kappa} \xi_{KL} \right\}. \tag{2.13}$$

An elastic–plastic material is said to be hardening, softening on exhibiting perfectly plastic behavior according to whether $\Phi$ is positive, negative or zero, respectively. During loading, the function $\Phi$ has the same value as the quotient $\hat{f}/\hat{g}$ of the quantities defined in (2.8).

The quotient $\hat{f}/\hat{g}$ is a measure of the ratio of the outward velocities with which the yield surfaces in stress space and strain space are moving during plastic flow. While during loading the yield surface in strain space is always moving outwards locally, the corresponding yield surface in stress space may concurrently be moving outwards, inwards or may be stationary depending on whether the material is exhibiting hardening, softening or perfectly plastic behavior[8]. For the case of uniaxial loading of special elastic–plastic materials, the quotient $\hat{f}/\hat{g}$ is related to the slope of the stress–strain curve [8]. Traditionally, hardening corresponds to a positive value of the slope and perfectly plastic behavior to a zero value. Softening corresponds to a negative slope, such as occurs on the falling portion of the engineering stress versus engineering strain curve for uniaxial tension.

Since $\lambda\gamma^* > 0$, it is clear from (2.12) that hardening, softening and perfectly plastic behavior corresponds to $\Gamma$ being positive, negative and zero, respectively. In the present paper, we confine attention to hardening behavior only, so that $\Gamma > 0$. In this case the flow rule in (2.9d) can be written as

$$\dot{e}^p_{KL} = \frac{\hat{f}}{\Gamma}\frac{\partial f}{\partial s_{KL}} \neq 0, \tag{2.14}$$

where (2.12) has been used.

## 3. BEHAVIOR OF A SPECIAL CLASS OF ELASTIC–PLASTIC MATERIALS UNDER COMBINED UNIAXIAL LOADING AND PRESSURE

In this section we consider the response in small deformation of metals whose behavior is characterized by a simple set of constitutive equations appropriate for elastic–plastic materials which are homogeneous and initially isotropic in their reference configuration.

Recall that the infinitesimal elastic strain tensor is defined by

$$e^e_{KL} = e_{KL} - e^p_{KL}. \tag{3.1}$$

It is convenient to utilize the standard decomposition of tensors into their spherical and deviatoric (traceless) parts. Thus, for example, in the case of the strain tensor, we have

$$e_{KL} = \bar{e}\delta_{KL} + \gamma_{KL},\ \bar{e} = \frac{1}{3}e_{KK}, \tag{3.2}$$

where $\bar{e}\delta_{KL}$ is the spherical part of $e_{KL}$, $\gamma_{KL}$ is the deviatoric part of $e_{KL}$ and $\bar{e}$ is the mean normal strain. In a similar manner, we decompose $s_{KL}$, $e^e_{KL}$, $e^p_{KL}$ into spherical parts $\bar{s}\delta_{KL}$, $\bar{e}^e\delta_{KL}$, $\bar{e}^p\delta_{KL}$ and deviatoric parts $\tau_{KL}$, $\gamma^e_{KL}$, $\gamma^p_{KL}$, respectively.

The stress response in (2.6) is assumed to be given by generalized Hooke's law, so that

$$\bar{s} = 3k\,\bar{e}^e,\ \tau_{KL} = 2\mu\,\gamma^e_{KL}, \tag{3.3}$$

where $k(>0)$ and $\mu(>0)$ are, respectively, the bulk modulus and the shear modulus of elasticity. We will consider a special loading function of the form

$$\begin{aligned} f &= \tau_{KL}\tau_{KL} + 3\psi\left(\bar{s} - \frac{\eta}{3}\right)^2 - \kappa, \\ g &= 4\mu^2(\gamma_{KL} - \gamma^p_{KL})(\gamma_{KL} - \gamma^p_{KL}) + 3\psi\left[3k(\bar{e} - \bar{e}^p) - \frac{\eta}{3}\right]^2 - \kappa, \end{aligned} \tag{3.4}$$

where $\psi$ and $\eta$ are constants, and where (3.3) have been used to derive $g$ from $f$.

The loading function $(3.4)_1$ does not depend explicitly on plastic strain but includes a dependence on mean normal stress. It also allows for unequal yield strengths in tension and compression and, as we shall see, is therefore capable of describing an S-D effect. When $\psi = 0$ and $\kappa =$ constant, $(3.4)_1$ reduces to the usual von Mises yield function. If $\eta = 0$, (3.4) reduces to a loading function previously employed by Casey and Naghdi ([8], eqn 55) and by Green and Naghdi[10]. If $\psi = 1$, we note that $(3.4)_1$ may be written in the form

$$f = \left(s_{KL} - \frac{\eta}{3}\delta_{KL}\right)\left(s_{KL} - \frac{\eta}{3}\delta_{KL}\right) - \kappa. \tag{3.5}$$

The coefficient function $\xi_{KL}$ for the rate of work-hardening response in (2.10) is assumed to be of the form

$$\xi_{KL} = \beta\tau_{KL} + \phi\left(\bar{s} - \frac{\eta}{3}\right)\delta_{KL}, \tag{3.6}$$

where $\beta$ and $\phi$ are constants, which reduces to ([8], eqn 54) if $\eta = 0$.

It can be easily shown that

$$\begin{aligned}
\frac{\partial f}{\partial s_{KL}} &= 2\left[\tau_{KL} + \psi\left(\bar{s} - \frac{\eta}{3}\right)\delta_{KL}\right],\\
\hat{f} &= 2\left[\tau_{KL}\dot{\tau}_{KL} + 3\psi\left(\bar{s} - \frac{\eta}{3}\right)\dot{\bar{s}}\right],\\
\hat{g} &= 2\left[2\mu\tau_{KL}\dot{\gamma}_{KL} + 9\psi k\left(\bar{s} - \frac{\eta}{3}\right)\dot{\bar{e}}\right]\\
&= \hat{f} + 2\left[2\mu\tau_{KL}\dot{\gamma}^p_{KL} + 9\psi k\left(\bar{s} - \frac{\eta}{3}\right)\dot{\bar{e}}^p\right].
\end{aligned} \tag{3.7}$$

During loading, $f = \hat{f} = 0$, so that

$$\begin{aligned}
f &= \tau_{KL}\tau_{KL} + 3\psi\left(\bar{s} - \frac{\eta}{3}\right)^2 - \kappa = 0,\\
&2\left[\tau_{KL}\dot{\tau}_{KL} + 3\psi\left(\bar{s} - \frac{\eta}{3}\right)\dot{\bar{s}}\right] - \dot{\kappa} = 0, \hat{f} = \dot{\kappa},
\end{aligned} \tag{3.8}$$

and hence for hardening behavior

$$\dot{\kappa} \text{ and } \left[\tau_{KL}\dot{\tau}_{KL} + 3\psi\left(\bar{s} - \frac{\eta}{3}\right)\dot{\bar{s}}\right] \text{ are both positive.} \tag{3.9}$$

Also, using (3.4), (3.6) and $(3.7)_1$, from (2.8) and (2.13) we obtain

$$\begin{aligned}
\Lambda &= 4\left[2\mu\tau_{KL}\tau_{KL} + 9\psi^2 k\left(\bar{s} - \frac{\eta}{3}\right)^2\right] > 0,\\
\Gamma &= 2\left[\beta\tau_{KL}\tau_{KL} + 3\psi\phi\left(\bar{s} - \frac{\eta}{3}\right)^2\right] > 0.
\end{aligned} \tag{3.10}$$

We now consider a homogeneous motion of a right circular cylindrical body sustained by a combination of applied uniform pressure $p^* \geqslant 0$, and uniaxial loading $s^*$ which is uniformly distributed on the bases of the cylinder. Both $p^*$ ans $s^*$ are functions of time only. For tension $s^* \geq 0$, while $s^* \leq 0$ for compression.

Let the initial length and radius of the cylinder be $l_0$ and $r_0$, respectively, and the deformed length and radius be $l$ and $r$. Assume an axially symmetric motion of the form

$$x_1 = (1+\lambda_1)X_1,\ x_2 = (1+\lambda_2)X_2,\ x_3 = (1+\lambda_2)X_3, \tag{3.11}$$

where $\lambda_1$, $\lambda_2$ are functions of time only, with initial values equal to zero. The deformation gradient (in matrix form), its determinant and the Lagrangian strain in the motion (3.11) are given by

$$\begin{aligned} \|F_{iA}\| &= \begin{Vmatrix} 1+\lambda_1 & 0 & 0 \\ 0 & 1+\lambda_2 & 0 \\ 0 & 0 & 1+\lambda_2 \end{Vmatrix}, \\ J &= (1+\lambda_1)(1+\lambda_2)^2, \\ \|e_{KL}\| &= \frac{1}{2}\begin{Vmatrix} (1+\lambda_1)^2-1 & 0 & 0 \\ 0 & (1+\lambda_2)^2-1 & 0 \\ 0 & 0 & (1+\lambda_2)^2-1 \end{Vmatrix}, \end{aligned} \tag{3.12}$$

where (2.2) and (2.3) have been used.

The Cauchy stress tensor is of the form

$$\|t_{ij}\| = -p^*\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + s^*\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}. \tag{3.13}$$

Therefore, in view of (2.4), $(3.12)_{1,2}$ and (3.13), the symmetric Piola–Kirchhoff stress tensor may be written as

$$\|s_{KL}\| = -p\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + s\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \tag{3.14}$$

where

$$\begin{aligned} p &= p^*(1+\lambda_1), \\ s &= p^*(1+\lambda_1)\left\{1-\left(\frac{1+\lambda_2}{1+\lambda_1}\right)^2\right\} + s^*\frac{(1+\lambda_2)^2}{1+\lambda_1}. \end{aligned} \tag{3.15}$$

From (3.14) we calculate

$$\bar{s} = \frac{s}{3} - p,\ \tau_{KL} = \frac{s}{3}b_{KL}, \tag{3.16}$$

where the matrix

$$\|b_{KL}\| = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \tag{3.17}$$

has been introduced for convenience. During loading it follows from $(3.8)_1$ and (3.16) that

$$f = \frac{1}{3}[2s^2 + \psi(s-3p-\eta)^2] - \kappa = 0. \tag{3.18}$$

For a fixed value of $\kappa$, (3.18) describes an ellipse in the $p$–$s$ plane if and only if $\psi > 0$. By introducing the change of variables

$$\begin{aligned} s'' &= s\cos\theta - p'\sin\theta, \\ p'' &= s\sin\theta + p'\cos\theta, \\ p' &= p + \frac{\eta}{3},\ \tan 2\theta = \frac{3\psi}{1-4\psi}, \end{aligned} \tag{3.19}$$

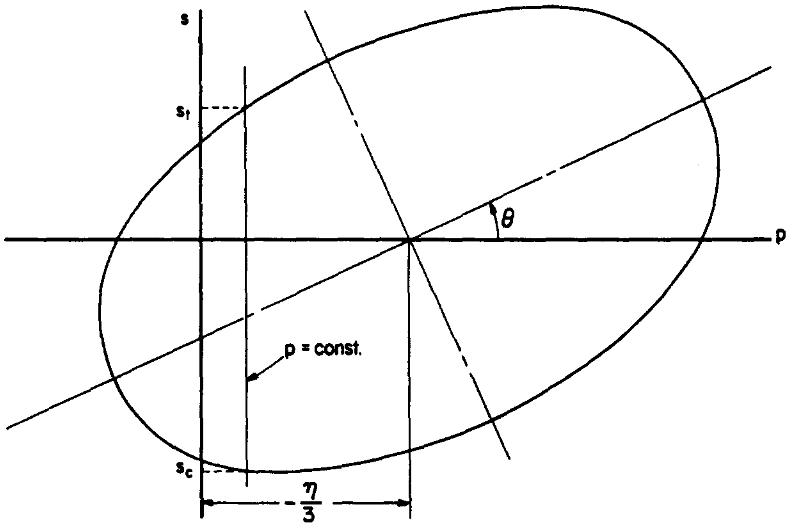



Fig. 1. Sketch of the yield surface described by (3.18) or (3.20). At a constant value of $p$, the material yields at a tensile stress $s_t$ and a compressive stress $s_c$.

(3.18) may be reduced to the form

$$\{1 + 5\psi - (1 - 8\psi + 25\psi^2)^{\frac{1}{2}}\}p''^2 + \{1 + 5\psi + (1 - 8\psi + 25\psi^2)^{\frac{1}{2}}\}s''^2 = 3\kappa. \quad (3.20)$$

In Fig. 1, an ellipse described by (3.20) is drawn for a fixed value of $\kappa$.

For hardening behavior we have

$$\begin{aligned} \hat{f} = \dot{\kappa} &= \frac{2}{3}[2s\dot{s} + \psi(s - 3p - \eta)(\dot{s} - 3\dot{p})] > 0, \\ \Lambda &= \frac{4}{3}[4\mu s^2 + 3\psi^2 k(s - 3p - \eta)^2] > 0, \\ \Gamma &= \frac{2}{3}[2\beta s^2 + \psi\phi(s - 3p - \eta)^2] > 0, \end{aligned} \quad (3.21)$$

where (3.16), $(3.8)_3$, (3.9) and (3.10) have been used.

For the rate of plastic strain during hardening behavior, we find with the use of (2.14), $(3.7)_1$, (3.16), and $(3.21)_{1,3}$ that

$$\begin{aligned} \dot{e}^p &= \frac{2}{3}\psi(s - 3p - \eta)\frac{2s\dot{s} + \psi(s - 3p - \eta)(\dot{s} - 3\dot{p})}{2\beta s^2 + \psi\phi(s - 3p - \eta)^2}, \\ \dot{\gamma}^p_{KL} &= \frac{2s}{3}\,\frac{2s\dot{s} + \psi(s - 3p - \eta)(\dot{s} - 3\dot{p})}{2\beta s^2 + \psi\phi(s - 3p - \eta)^2}b_{KL}. \end{aligned} \quad (3.22)$$

In Section 4, we will compare the results of the present section with experimental data obtained by Spitzig *et al.* [2]. In anticipation of this, we consider now the special case for which $\dot{p} = 0$. Thus, from (3.22) we obtain

$$\begin{aligned} \dot{e}^p &= \frac{\dot{s}}{9k^*(s, p)}, \quad \dot{\gamma}^p_{KL} = \frac{\dot{s}}{6\mu^*(s, p)}b_{KL}, \\ \|\dot{e}^p_{KL}\| &= \frac{\dot{s}}{E^*(s, p)}\begin{Vmatrix} 1 & 0 & 0 \\ 0 & -\nu^*(s, p) & 0 \\ 0 & 0 & -\nu^*(s, p) \end{Vmatrix}, \end{aligned} \quad (3.23)$$

where

$$\nu^*(s, p) = \frac{s - \psi(s - 3p - \eta)}{2s + \psi(s - 3p - \eta)},$$

$$E^*(s, p) = \frac{3}{2}\,\frac{2\beta s^2 + \psi\phi(s - 3p - \eta)^2}{[2s + \psi(s - 3p - \eta)]^2} > 0, \tag{3.24}$$

$$\mu^*(s,p) = \frac{E^*(s,p)}{2[1 + \nu^*(s,p)]},\quad k^*(s, p) = \frac{E^*(s, p)}{3[1 - 2\nu^*(s, p)]},$$

and the positivity of $E^*(s, p)$ follows from $(3.21)_3$. The material coefficients in (3.24) have been defined as in Casey and Naghdi ([8], Section 4) and are analogous to the constants of linear elasticity. The elastic strains are

$$\bar{e}^e = \frac{s - 3p}{9k},\quad \gamma^e_{KL} = \frac{s}{6\mu} b_{KL}. \tag{3.25}$$

During loading $\dot{\kappa} > 0$ and therefore, at constant $p$, $\dot{s}$ is nonzero and we can obtain the derivatives

$$\frac{d\bar{e}}{ds} = \frac{d\bar{e}^e}{ds} + \frac{\dot{\bar{e}}^p}{\dot{s}} = \frac{1}{9}\left\{\frac{1}{k} + \frac{1}{k^*(s, p)}\right\},$$

$$\frac{d\gamma_{KL}}{ds} = \frac{1}{6}\left\{\frac{1}{\mu} + \frac{1}{\mu^*(s, p)}\right\} b_{KL}, \tag{3.26}$$

$$\left\|\frac{de_{KL}}{ds}\right\| = \frac{1}{E}\begin{Vmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{Vmatrix} + \frac{1}{E^*(s,p)}\begin{Vmatrix} 1 & 0 & 0 \\ 0 & -\nu^*(s, p) & 0 \\ 0 & 0 & -\nu^*(s, p) \end{Vmatrix},$$

$$\frac{d\kappa}{ds} = \frac{2}{3}[(2 + \psi)s - \psi(3p + \eta)],$$

where (3.25), (3.23) and $(3.21)_1$ have been used.

In a region of hardening, we may write (2.12) as $\Phi = (1 + \Lambda/\Gamma)^{-1}$. Then, by $(3.21)_{2,3}$ and $(3.24)_{1,2}$

$$\begin{aligned}\Phi &= \left[1 + \frac{E}{3E^*(s, p)}\left\{\frac{2(1 + \nu^*(s, p))^2}{1 + \nu} + \frac{(1 - 2\nu^*(s, p))^2}{1 - 2\nu}\right\}\right]^{-1} \\ &= \left[1 + \frac{E}{E^*(s, p)}\left\{1 + \frac{2(\nu - \nu^*(s, p))^2}{(1 + \nu)(1 - 2\nu)}\right\}\right]^{-1},\end{aligned} \tag{3.27}$$

where $E$ is Young's modulus and $\nu$ is Poisson's ratio. Equations of the form (3.27) were previously derived in ([8], Appendix).

Finally, we provide a definition of the strength-differential effect at constant $p$. If at any given $p$ and for fixed values of plastic strain and $\kappa$, the material yields at a tensile stress $s_t > 0$, and at a compressive stress $s_c < 0$ (see Fig. 1), the S-D effect is

$$\text{S-D} = 2\,\frac{|s_c| - |s_t|}{|s_c| + |s_t|} = 2\,\frac{s_c + s_t}{s_c - s_t}. \tag{3.28}$$

But, from (3.18)

$$\begin{aligned}(2 + \psi)s_c &= \psi(3p + \eta) - [3(2 + \psi)\kappa - 2\psi(3p + \eta)^2]^{1/2}, \\ (2 + \psi)s_t &= \psi(3p + \eta) + [3(2 + \psi)\kappa - 2\psi(3p + \eta)^2]^{1/2}.\end{aligned} \tag{3.29}$$

Therefore,

$$s_c + s_t = \frac{2\psi(3p+\eta)}{2+\psi},$$
$$s_c\, s_t = \frac{\psi(3p+\eta)^2 - 3\kappa}{2+\psi}. \tag{3.30}$$

Also,

$$\text{S-D} = -\frac{2\psi(3p+\eta)}{[3(2+\psi)\kappa - 2\psi(3p+\eta)^2]^{1/2}}. \tag{3.31}$$

## 4. DETERMINATION OF THE MATERIAL COEFFICIENTS

In this section we use the experimental data of Spitzig *et al.* [2] for AISI 4330 steel to determine the material coefficients $\eta$, $\psi$, $\beta$ and $\phi$, as well as other quantities from Section 3. In order to convert the data of [2] to a form that can be used in our equations, we record the relationship between the strain measure used above and that of [2]. In the motion (3.11), the deformed and undeformed dimensions of the cylindrical body are related by

$$l = (1+\lambda_1)l_0,\quad r = (1+\lambda_2)r_0. \tag{4.1}$$

The "true strain" tensor is then given by

$$\|\epsilon_{AB}\| = \begin{Vmatrix} \ln(l/l_0) & 0 & 0 \\ 0 & \ln(r/r_0) & 0 \\ 0 & 0 & \ln(r/r_0) \end{Vmatrix} = \begin{Vmatrix} \ln(1+\lambda_1) & 0 & 0 \\ 0 & \ln(1+\lambda_2) & 0 \\ 0 & 0 & \ln(1+\lambda_2) \end{Vmatrix}. \tag{4.2}$$

It follows from (4.1), (4.2) and $(3.12)_3$ that

$$2e_{11} = \exp(2\epsilon_{11}) - 1$$
$$2e_{22} = 2e_{33} = \exp(2\epsilon_{22}) - 1. \tag{4.3}$$

The maximum value of $|\epsilon_{11}|$ encountered in the experiments of [2] is about 0.04, so we will assume that in expressions involving strain, terms of second order in the true strain may be neglected in comparison with linear terms. Then,

$$e_{11} = \epsilon_{11} = \lambda_1,\quad e_{22} = e_{33} = \epsilon_{22} = \lambda_2, \tag{4.4}$$

approximately.

Next, we recall that the relationship between an element of volume $dv_0$ in the reference configuration and its image $dv$ in the deformed configuration is

$$dv = J\, dv_0. \tag{4.5}$$

To the order of approximation being considered

$$J = 1 + e_{KK}. \tag{4.6}$$

Similarly, the plastic volume element $dv^p$ is given by

$$dv^p = (1 + e^p_{KK})\, dv_0. \tag{4.7}$$

Hence, the rate of plastic volume change per unit initial volume is

$$\frac{\overline{\mathrm{d}v^p}^{\,\cdot}}{\mathrm{d}v_0} = \dot{e}^p_{KK}, \tag{4.8}$$

and the ratio

$$\gamma^p = \frac{\overline{\mathrm{d}v^p}^{\,\cdot}}{\mathrm{d}v_0}\bigg/\dot{e}^p_{11} \tag{4.9}$$

may be expressed as

$$\gamma^p = \frac{\mathrm{d}e^p_{KK}}{\mathrm{d}e^p_{11}} = 1 - 2\nu^*(s, p), \tag{4.10}$$

where $(3.23)_3$ has been used.

For AISI 4330 steel, Young's modulus and Poisson's ratio have values $E = 28700$ (ksi) and $\nu = 0.29$, respectively [2]. Spitzig *et al.*[2] measured plastic volume change and plotted their results in Fig. 7 of [2]. The data points are scattered considerately, especially in the region of initial yield, but lie approximately on a straight line of slope $|\gamma^p| = 0.005$, which in view of (4.10) corresponds to a value of 0.4975 for $\nu^*(s, p)$ in tension and 0.5025 in compression. Using these values† and appealing to (3.25) and (4.4), (3.15) may be written in the form

$$\begin{aligned} p &= \alpha_1 p^*, \\ s &= \alpha_2 s^* + (\alpha_1 - \alpha_2) p^*, \end{aligned} \tag{4.11}$$

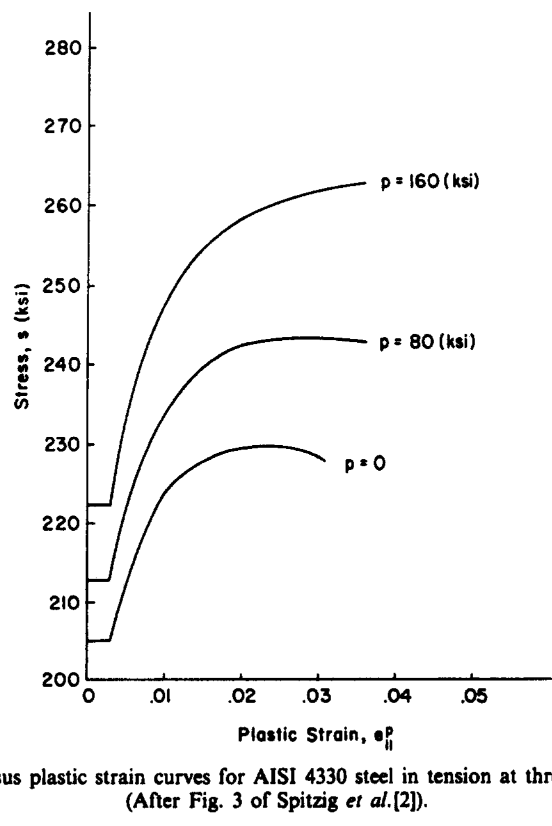



Fig. 2. Stress versus plastic strain curves for AISI 4330 steel in tension at three fixed pressures. (After Fig. 3 of Spitzig *et al.*[2]).

†As we shall see below, other data from [2], when used in our equations lead to somewhat different values for $\nu^*(s, p)$. The present calculation is not very sensitive to these differences.

where $\alpha_1$ and $\alpha_2$ are given by the approximations†

$$\alpha_1 = \begin{cases} 1.009 + \epsilon_{11}^p, \text{(tension)}, \\ 0.991 + \epsilon_{11}^p, \text{(compression)}, \end{cases}$$

$$\alpha_2 = \begin{cases} 0.986 - 1.995\epsilon_{11}^p, \text{(tension)} \\ 1.014 - 2.006\epsilon_{11}^p, \text{(compression)}. \end{cases}$$

Data from Figs. 3 and 4 of Spitzig *et al.* [2] were substituted in (4.12) and (4.11) in order to obtain the corresponding values of $p$ and $s$. Also, since $p^*$ rather than $p$ was held constant in the experiments of [2], a small correction was applied to account for the change in $p$. However, in the worst case in tension, $p$ increases from 160 (ksi) to 167 (ksi) at $\epsilon_{11}^p = 0.035$ and the corresponding correction to $s^*$ is only 1 (ksi). The converted data are plotted in Figs. 2 and 3.

Next, we consider the variation of yield stress $s$ with respect to $p$ at constant values of plastic strain and $\kappa$. Thus, we use (3.18) and (3.24)$_1$ to calculate the partial derivative

$$\frac{\partial s}{\partial p} = \frac{3\psi(s - 3p - \eta)}{2s + \psi(s - 3p - \eta)} = 1 - 2\nu^*(s, p). \tag{4.13}$$

We further note that

$$\frac{\partial s}{\partial p} = \frac{\partial f}{\partial s_{KK}} \Big/ \frac{\partial f}{\partial s_{11}} = \dot{e}_{KK}^p / \dot{e}_{11}^p, \tag{4.14}$$

where (4.13), (3.7)$_1$, (3.16), and (2.14) have been used. From (4.13) and (4.10), it is clear

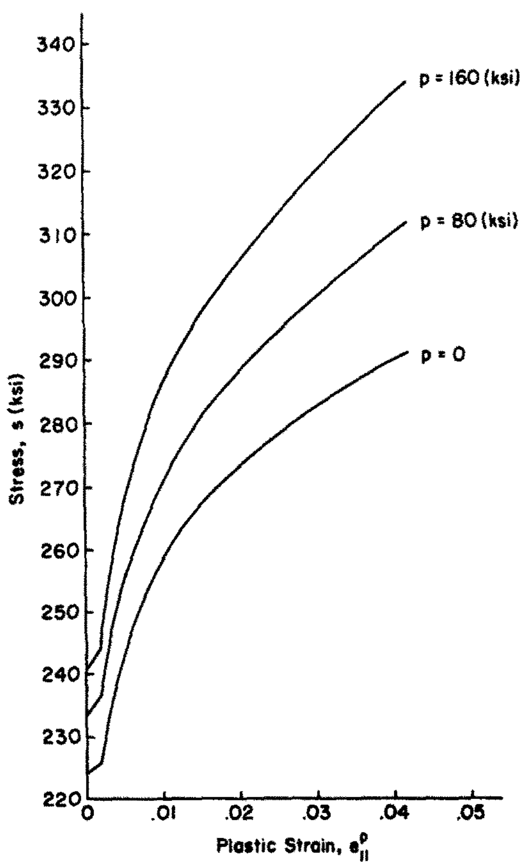



Fig. 3. Stress versus plastic strain curves for AISI 4330 steel in compression at three fixed pressures. (After Fig. 4 of Spitzig *et al.* [2]).

†The magnitude of the elastic strain $\epsilon_{11}^e$ is almost constant, at a value of 0.009.

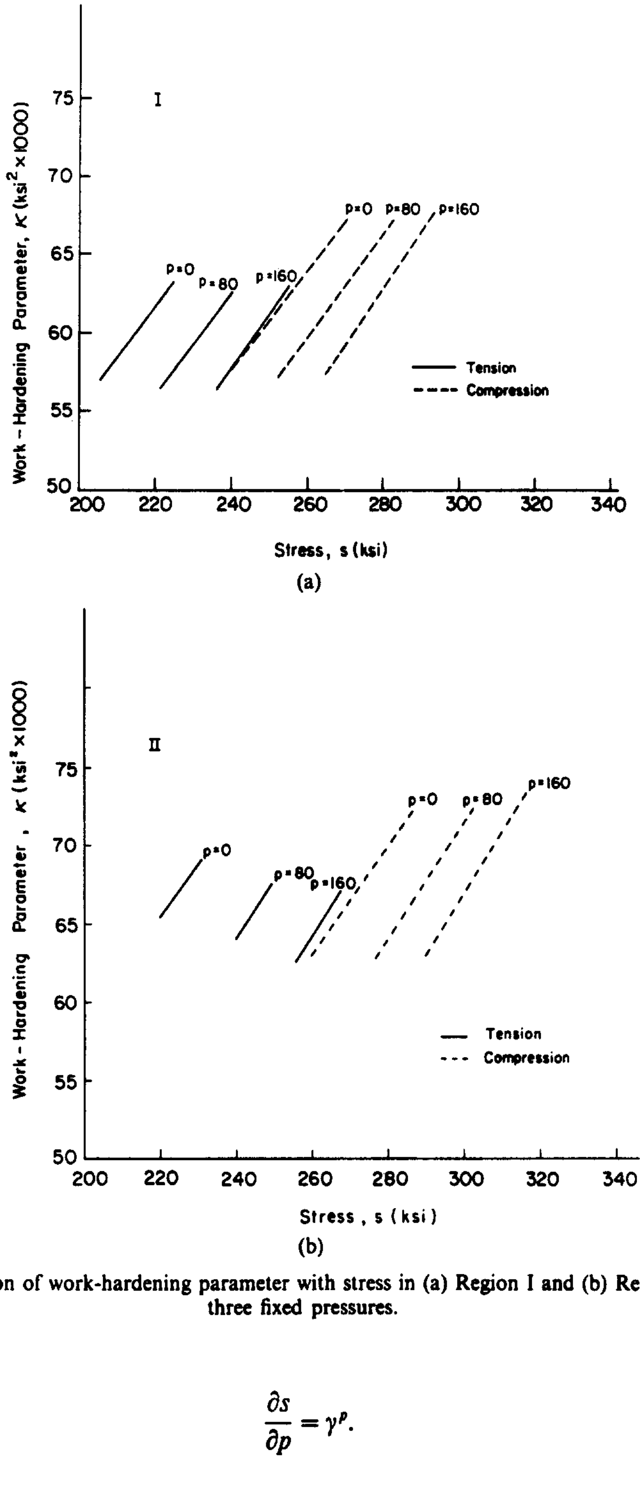



Fig. 4. Variation of work-hardening parameter with stress in (a) Region I and (b) Region II, for three fixed pressures.

that

$$\frac{\partial s}{\partial p} = \gamma^p. \tag{4.15}$$

In order to obtain values for $\eta$ and $\psi$, we used a trial and error method to obtain solutions to (3.30) which satisfied the data in Figs. 2 and 3. (A first approximation is obtained by neglecting $\psi$ in comparison to 2.) Rather than find values of $\psi, \eta$ which would give a reasonable approximation over the entire range of plastic strain in Figs. 2 and 3, we obtained values which separately gave good approximations along the lower portions of the curves (Region I) and along the upper portions (Region II). The calculated values of $\psi$, $\eta$ are listed in Table 1. The variation of $\kappa$ as a function of $s$, obtained from (3.18), is plotted in Fig. 4. Typical yield surfaces for the two regions are shown in Fig. 5. The variation of $v^*(s, p)$, as given by $(3.24)_1$, is presented graphically in Fig. 6. The strength-differential effect, obtained from (3.31), is plotted as a function of $\kappa$ in Fig. 7.

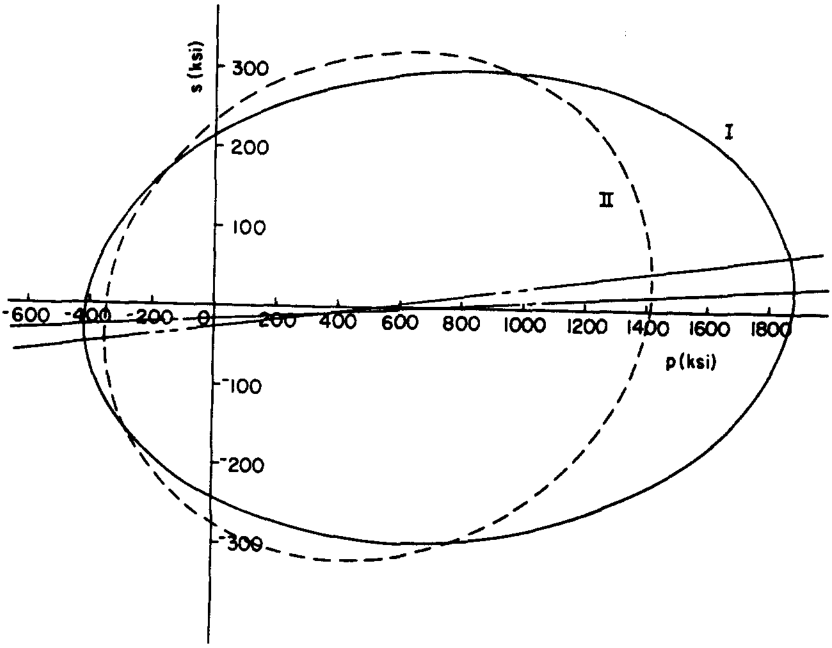
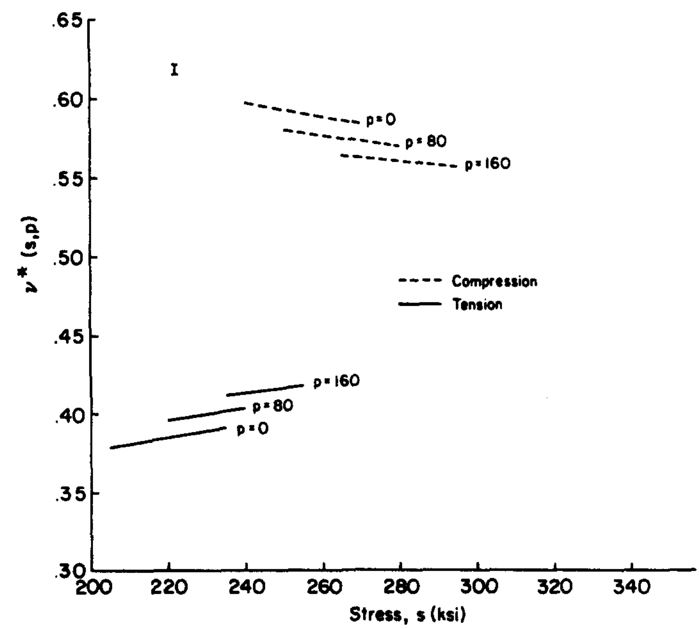



Fig. 5. Typical yield surfaces in Regions I and II. For the surface in Region I, $\kappa = 59000$ (ksi$^2$) and $\theta = 1.37°$. For the surface in Region II, $\kappa = 69000$ (ksi$^2$) and $\theta = 2.92°$.

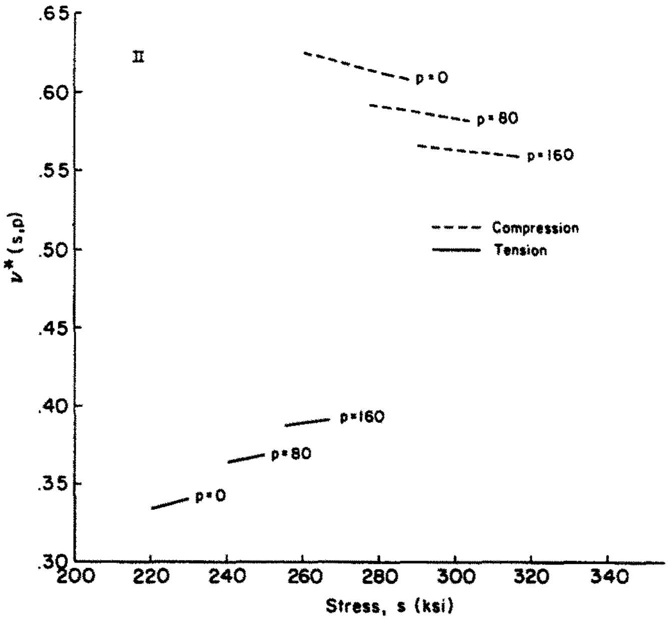
To obtain values of $\beta$ and $\phi$, measured values of the slopes of Figs. 2 and 3 were used in conjunction with $(3.26)_1$ and $(3.24)_2$. However, it is difficult to determine these constants accurately by this method. Approximate values are given in Table 1. Corresponding predicted values of $E^*(s, p)$, as given by $(3.24)_2$, may also be found in Table. 1. In order to quantify the strain-hardening behavior, we used (3.27) to calculate $\Phi$. The resulting values are given in Table 1 as well.

Due to characteristic differences between the curves for uniaxial tension and compression in the region of initial yield (see Figs. 2 and 3 in the range 0–0.003 for plastic strain), we also calculated values of certain material coefficients based specifically on the data for this region. The results are recorded in Table 2. However, as mentioned in the




(a)

Fig. 6.




(b)

Fig. 6. Variation of $\nu^*(s, p)$ with stress in (a) Region I and (b) Region II, at three fixed pressures.

Introduction, it is the behavior of the material over the entire plastic range—rather than its behavior in the region of initial yield—that is of importance in discussing the S–D effect.

## 5. DISCUSSION

In this section we discuss certain features of the results contained in Sections 3 and 4.

To summarize, a yield function (3.4) was chosen which, for the type of loading used in the experiments of [2], reduces to the form (3.18). The stress response was assumed to

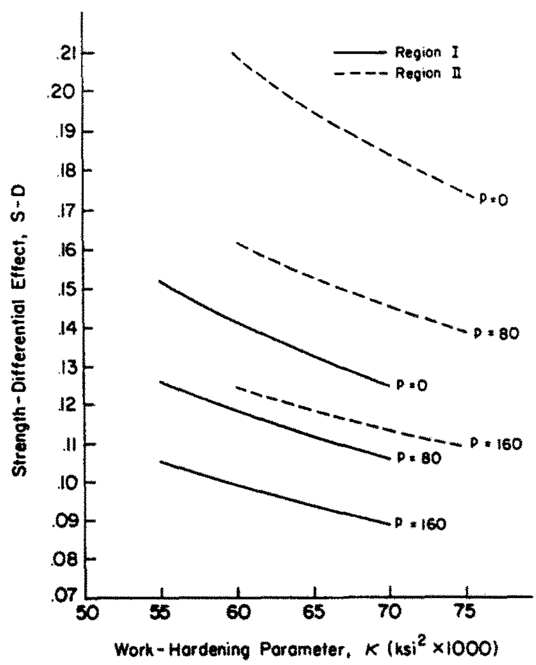



Fig. 7. The strength-differential effect versus the work-hardening parameter in Regions I and II at three fixed pressures.

Table 1. Values of various functions and material coefficients for Regions I and II

| Region | $p$(ksi) | Tension or Compression | $\Phi$ | $E^*(s, p)$(ksi) | $\phi$(ksi) | $\beta$(ksi) | $\eta$(ksi) | $\psi$ |
|---|---|---|---|---|---|---|---|---|
| I | 0 | T | 0.100 | 3280 | −4400 | 9500 | −2200 | 0.015 |
| | | C | 0.136 | 6140 | | | | |
| | 80 | T | 0.119 | 4050 | | | | |
| | | C | 0.149 | 6590 | | | | |
| | 160 | T | 0.135 | 4720 | | | | |
| | | C | 0.158 | 6890 | | | | |
| II | 0 | T | (−0.010) | (−291) | −2500 | 1900 | −1600 | 0.03 |
| | | C | 0.022 | 904 | | | | |
| | 80 | T | 0.006 | 165 | | | | |
| | | C | 0.030 | 1170 | | | | |
| | 160 | T | 0.018 | 542 | | | | |
| | | C | 0.035 | 1330 | | | | |

Table 2. Values of various functions and material coefficients for the region of initial yield

| $p$(ksi) | Tension or compression | $v^*(s,p)$ | $S$–$D$ | $\theta^\circ$ | $\kappa$(ksi$^2$) | $\psi$ | $\eta$(ksi) |
|---|---|---|---|---|---|---|---|
| 0 | T | 0.419 | 0.099 | | | | |
| | C | 0.570 | | | | | |
| 80 | T | 0.428 | 0.086 | 0.664 | 53000 | 0.0075 | −2900 |
| | C | 0.560 | | | | | |
| 160 | T | 0.436 | 0.075 | | | | |
| | C | 0.552 | | | | | |

obey (3.3) and the constitutive equation for plastic strain rate is given by (2.14) during hardening behavior. The strength-differential effect is given by (3.31), and the ratio $\gamma^p$ associated with plastic volume change, by (4.10). The quantities listed in Tables 1 and 2 and those plotted in Figs. 2–7 were determined using data from[2].

Clearly, the yield function (3.18) predicts both a change in plastic volume and a strength-differential effect. We observe that if $\psi = 0$, then by $(3.24)_1$ and (4.11), $v^* = 1/2$ and $\gamma^p = 0$. Furthermore, by (3.31), S–D = 0 also. Therefore, in order to describe both change in plastic volume and a strength-differential effect it is necessary to include a dependence on mean normal stress $\bar{s}$ in the yield function (3.4). However, it is also worth emphasizing that unless $\eta \neq 0$ as well, no S–D effect is predicted at $p = 0$.

Also in regard to the yield function in the form (3.18), we note that for fixed values of $s$-$3p$, (3.18) gives the same yield stress in tension and in compression. This featųre agrees with the data of Spitzig *et al.*[2], as observed by Gupta[3].

For the range of loading covered in the experiments of [2], we see from Fig. 4 that the work-hardening parameter $\kappa$ is virtually linear as a function of stress both in Regions I and II. Thus, the slope in $(3.26)_4$ is virtually constant for a given region and pressure.

From Fig. 5, it is clear that the effect of plastic deformation is to move the center of the ellipse closer to the origin in stress space and to increase the tilt of the axis. For small $\psi$, the ellipses in Fig. 5 are given with sufficient accuracy by the following approximations to (3.19) and (3.20):

$$9\psi p''^2 + (2+\psi)s''^2 = 3\kappa, \quad \theta = \frac{3}{2}\psi \quad \text{(radian measure)}$$
$$p'' = s\theta + p', \quad s'' = s - p'\theta, \tag{5.1}$$

so that the semimajor and semiminor axes of the ellipses have approximate lengths $\{\kappa/3\psi\}^{1/2}$ and $\{3\kappa/(2+\psi)\}^{1/2}$, respectively.

In Fig. 6 we see that the relationship between $v^*(s,p)$ and $s$ at constant $p$ does not deviate noticeably from linearity in either region. We observe, however, that for tension, $v^*(s,p)$ is less than 1/2, while for compression $v^*(s,p)$ is greater than 1/2. As $p$ increases, $v^*(s,p)$ tends towards 1/2 in both cases. As mentioned before, Spitzig *et al.*[2] report that $\gamma^p = 0.005$ approximately for tension (and the negative of this value for compression). This corresponds values of 0.4975 and 0.5025 for $v^*(s,p)$ according to (4.10). In Region I, at $p = 0$, typical values for $v^*(s,p)$ according to the present theory are (see Fig. 6a) 0.385 for tension and 0.590 for compression. These correspond to $\gamma^p = 0.23$ for tension and $\gamma^p = -0.20$ for compression.

Spitzig *et al.*[2] also found a large difference between the value predicted for $\gamma^p$ from their linear yield function and the measured value. These authors suggested that the normality condition for plastic strain rate does not hold.† This possibility will not be pursued in the present paper.

†See also the later investigation[11] by Spitzig *et al.* on maraging and HY-80 steels.

The calculated strength-differential effect which is plotted in Fig. 7 gives good agreement with the data in Figs. 2 and 3.

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